Stochastic process and Markov chains

Stochastic is a Greek word which means random or chance

Def :- Mathematically a stochastic process is a set of random variables $\{x(t)\}$ depending on some real parameter like time t. ~~sam(t)~~. ~~small(t)~~. These are also known as random processes or Random functions.

eg:- consider x_n as the max no. shown in the first n throws. we can see that $\{x_n / n \geq 1\}$ constitutes a stochastic process. state : It is a condition or) location of an object in the system at particular time.

Stochastic Matrix

→ It is a square matrix

→ And having non-negative entries

→ The sum of elements in each row is equal to 1.

(1) which of the following matrices are stochastic

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3}$

It is not a square matrix
so it is not stochastic matrix.

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

* It is a square matrix.
* all the elements are non-negative
* sum of each row equal to 1.

$\therefore$ It is a stochastic matrix.

(iii) $\begin{bmatrix} 0 & 1 \\ 1/3 & 1/4 \end{bmatrix}_{2\times 2}$

* It is a square matrix
* All the elements are non-negative
* sum of each row not equal to 1

$\therefore$ It is not stochastic matrix

(iv) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $\therefore$ It is stochastic matrix

v) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}_{2\times 2}$ having negative elements

so $\therefore$ It is not stochastic matrix.

vi) $\begin{bmatrix} 0 & 2 \\ 1/4 & 1/4 \end{bmatrix}$ not stochastic matrix
sum of each row not equal to 1.

Regular matrix

A stochastic matrix P is said to be a regular matrix, if all entries of some power P^m are positive.

→ A stochastic matrix P is not regular matrix if '1' occurs in the principle main diagonal.

→ Also P has no zero elements.

problems

(I) which of the following matrices are regular.

(i) $A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

'1' is lies on the principle diagonal so it is not a regular matrix.

(ii) $B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

$B^2 = B \cdot B \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} =$

$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 7/16 & 7/16 & 1/8 \end{bmatrix}$$

the entries B_{13} & B_{23} having zeros

.so it is not a regular matrix.

(iii) $$C = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 1/2 & 0 & 1 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$C^4 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}, \quad C^5 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 1/2 & 3/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

all entries of some power of C are +ve

∴ C is a regular stochastic matrix.

H.W

(1) $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ is Regular matrix ?

Transition

Transistion probability Matrix

The transistion ~~probability~~ : probabilities P_{ij} satisfies

i = initial
j = next state

(i) $P_{ij} \geq 0$ { all are +ve }

(ii) $\sum P_{ij} = 1, \forall i$

These probabilities may be written in the matrix form.

$$P = \begin{array}{c|cccc} & S_1 & S_2 & S_3 & \\ S_1 & P_{11} & P_{12} & P_{13} & \cdots \\ S_2 & P_{21} & P_{22} & P_{23} & \cdots \\ S_3 & P_{31} & P_{32} & P_{33} & \cdots \end{array}$$

This is called the transition probability matrix of the markov chain.

Markov chain :

The transition probability matrix P has defined with the initial probabilities $\{P_j^0\}$ associated with the state E_j where $E_j = 0, 1, 2 \cdots n$ completely define a markov chain.

Markov chains are 2 types

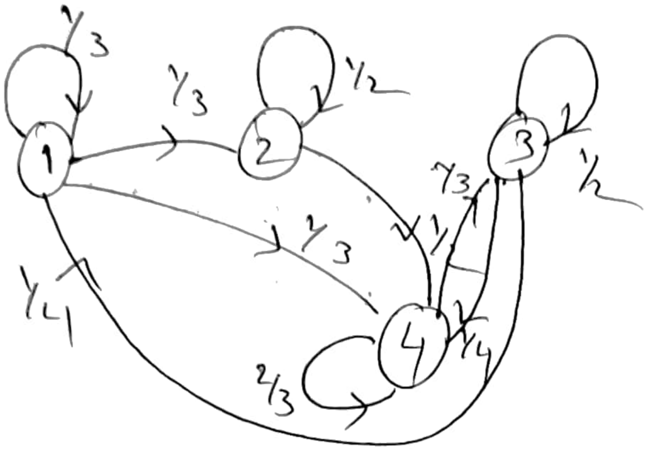
(1) Ergodic (2) Regular.

Ergodic : An ergodic markov chain the probability that it is possible to pass from one state to another state in finite no. of steps. Regarding of present state.

A regular Markov chain is defined as a chain having a transition matrix P such that for some power of P it has only non-zero positive probability values. thus all the regular chains must be ergodic.

(i) Determine if the following transition probability matrix is ergodic.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 4 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}$$

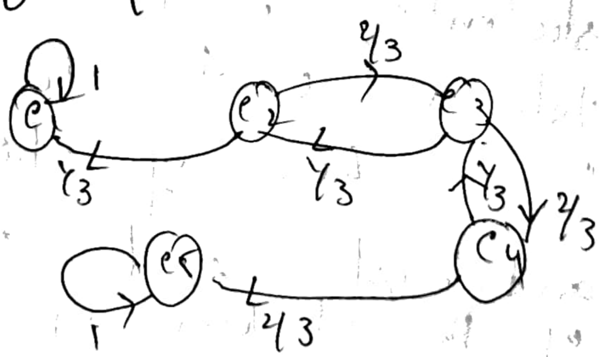


Here it is possible to go from every present state to all the other state.

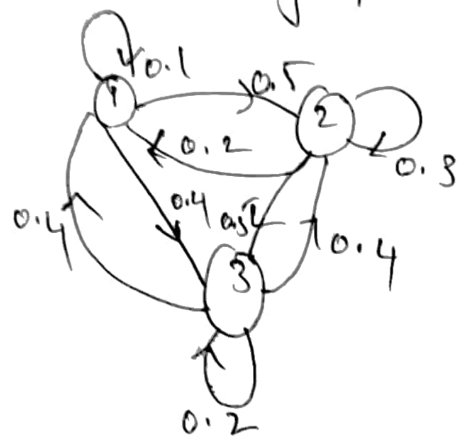
So the given TPM is ergodic markov chain

2) consider the TPM & find its graph.

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



(3) Draw graph to matrix



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.2 & 0.3 & 0.5 \\ 0.4 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

$\therefore$ All states are ergodic

Imp

(4) find the values of x, y, z if $\begin{bmatrix} 0 & x & 1/3 \\ 0 & 0 & y \\ 1/3 & 1/4 & z \end{bmatrix}$ is a TPM.

sol :- Sum of elements in each row = 1

$0 + x + 1/3 = 1 \Rightarrow x = 1 - 1/3 \Rightarrow \boxed{x = 2/3}$

$y = 1$

$1/3 + 1/4 + z = \cancel{0} \Rightarrow z = 1 - \frac{4+3}{12}$

$z = 1 - 7/12$

$z = \frac{12-7}{12}$

$\boxed{z = 5/12}$

(5) 3 universities A, B, C are admitting students. it is given that 80% of children went to A and the rest went to B. 40% of children of B went to B. and rest split evenly between A and C of the children of C. 70% went to C and

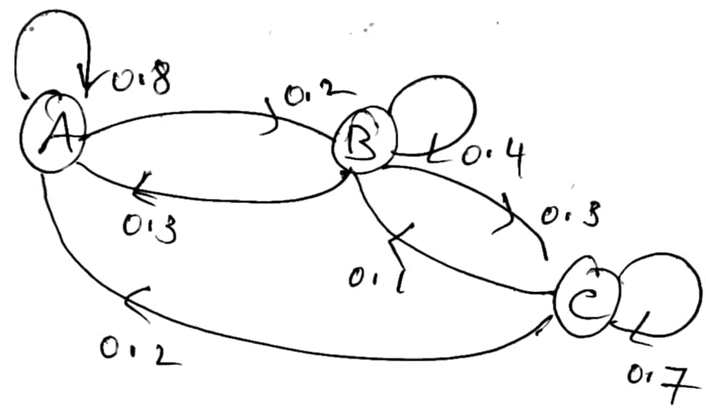
20% went to A and 10% to B (5)
~~70% went to C~~. find the markov chain and TPM.

Sol :- TPM :

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.8 & 0.2 & 0 \\ B & 0.3 & 0.4 & 0.3 \\ C & 0.2 & 0.1 & 0.7 \end{array}$$

This is the transition probability matrix.

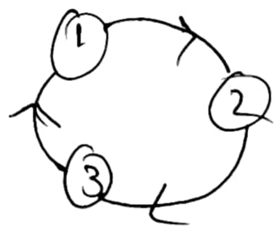
Markov chain :



Irreducible : A markov chain if all the states communicate to each other

ex :

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

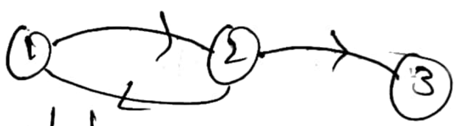


Recurrent state : come back to the s[illegible]
state.

Eg:-

state ① and state ② are called recurrent states.

Transiant state : Not come back.



the states ② & ③ are called Transiant state.

probabilities of Gambler Ruin

→ If $p=q$ then (i) probability of ruin $q_z = \frac{a-z}{a}$

(ii) expected duration of the game $d_z = z(a-z)$

→ If $p \neq q$ then (i) probability of ruin

$$q_z = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$$

(ii) expected duration of the game

$$d_z = \left(\frac{-a}{q-p}\right) \frac{\left(\frac{q}{p}\right)^z - 1}{\left(\frac{q}{p}\right)^a - 1} + \frac{z}{q-p}$$

(1) calculate the probability of ruin and expected duration of the game, where

(i) $a=50$, $z=40$, $p=0.5$

(ii) $a=100$, $z=5$, $p=0.6$

Sol:- (i) Given that $a=50$, $z=40$, $p=0.5$
$= 1/2$
$q = 1-p$
$= 0.5$

$\therefore$ probability of ruin $q_z = \frac{a-z}{a}$

$= \frac{50-40}{50} = \frac{10}{50}$

$= 0.2$

expected duration of the game $d_z = z(a-z)$

$= 40(50-40)$

$= 40 \times 10$

$= 400$

(ii) Given that $a=100$, $z=5$, $p=0.6$

$q = 1-p$

$= 1-0.6$

$q = 0.4$

$\frac{p}{q} = 2/3$

probability of ruin $q_z = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^z}{\left(\frac{q}{p}\right)^a - 1}$

$q_z = \frac{(2/3)^{100} - (2/3)^5}{(2/3)^{100} - 1} = 0.132$

expected duration of the game

$d_z = \left(\frac{-a}{q-p}\right) \frac{\left(\frac{q}{p}\right)^z - 1}{\left(\frac{q}{p}\right)^a - 1} + \frac{z}{q-p}$

$$= \frac{-100}{-0.2}\left(\frac{\left(\frac{2}{3}\right)^5 - 1}{\left(\frac{2}{3}\right)^{100} - 1}\right) + \frac{5}{-(0.2)} = 409$$

Higher transition probability

n step transition probability is given by $P(X_{m+n} = j / X_n = i) = P_{ij}^{(n)}$

eg:- $P\{X_4 = 2 / X_2 = 1\} = P_{12}^{(2)}$

① consider Markov chain

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

find $P_{01}^{(2)}$ and $P(X_2 = 1, X_0 = 0)$, $P(X_0 = 0) = 1/3$

Sol :-

$$P^2 = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix} \end{array}$$

$P_{01}^{(2)} = \frac{5}{16}$

$$P[X_2 = 1, X_0 = 0] = P[X_2 = 1 / X_0 = 0]\, P\{X_0 = 0\}\, P_{01}^{(2)}$$

$$= P_{01}^{(2)}\, P(X_0 = 0)$$

$$= 5/16 \cdot 1/3 = 5/48$$

(2) The transition probability matrix of a markov chain $\{x_n\}$ where $n = 1, 2, 3 \ldots$ having 3 states 1, 2 & 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ and initial distribution is $p^{(0)} = [0.7 \quad 0.2 \quad 0.1]$

find (i) $P(x_2 = 3)$

(ii) $P[x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2]$

Sol:- Given the transition probability matrix

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

Also given $P(x_0 = 1) = 0.7$

$P(x_0 = 2) = 0.2$

$P(x_0 = 3) = 0.1$

$$P^2 = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(i) $P(x_2 = 3) = \sum_{i=1}^{3} P\{x_2 = 3 / x_0 = i\} \cdot P\{x_0 = i\}$

$= P\{x_2 = 3 / x_0 = 1\} \cdot P\{x_0 = 1\} +$
$P\{x_2 = 3 / x_0 = 2\} \cdot P(x_0 = 2\} + P\{x_2 = 3 / x_0 = 3\}$
$P(x_0 = 3\}$

$= P_{13}^{(2)} P\{x_0 = 1\} + P_{23}^{(2)} P\{x_0 = 2\} + P_{33}^{(2)} P\{x_0 = 3\}$

$= 0.26\ (0.7) + 0.34\ (0.2) + 0.29\ (0.1)$

$= 0.279$

(ii) $P\{\cancel{x_1 = 3 / x_0 = 2}\} = \cancel{P_{23}^{0.2}}$

$P\{\cancel{x_1 = 3,}\ x_0 = 2\} = P\{x_1 = 3 / \cancel{x_0 = 2\} \cdot P\{x_0 = 2}\}$

$= \cancel{0.2 \times 0.2}$

$= \cancel{0.04}$

$P\{x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2\}$

$= P\{x_3 = 2 / x_2 = 3\} \cdot P\{x_2 = 3 / x_1 = 3\} \cdot P\{x_1 = 3 / x_0 = 2\}$
$\cdot P\{x_0 = 2\}$

$P_{32}^{(1)} * P_{33}^{(1)} \times P_{23}^{(1)}\ P\{x_0 = \}$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= 0.0048$

(3) A fair die is tossed repeatedly, if X_n denotes the maximum of the numbers occuring in the first n tosses, find the transition probability matrix P of the markov chain $\{X_n\}$. find also P^2 and $P(X_2=6)$

sol :- state space $= \{1, 2, 3, 4, 5, 6\}$

let X_n = the max of the numbers occuring in the first n trails = 3 say.

then $X_{n+1} = 3$, if the $(n+1)^{th}$ trail results is 1, 2 or 3.

$= 4$ if the $(n+1)^{th}$ trail results is 4.

$= 5$ if the $(n+1)^{th}$ trail results is 5

$= 6$ if the $(n+1)^{th}$ trail results is 6

The TPM is formed using the following analysis.

let X_n = the maximum of the numbers occuring in the first n trials = 3 (say)

then $X_{n+1} = 3$ if the $(n+1)^{th}$ trial results 1, 2, or 3

$= 4$ if the $(n+1)^{th}$ trial is 4

$= 5$ " " " " " 5

$= 6$ " " " " " 6

$\therefore P\{X_{n+1} = 3 / X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3/6 = 1/2$

$\therefore P\{X_{n+1} = i / X_n = i\} = 1/6$ where $i = 4, 5, 6$

$\therefore$ The TPM of chain is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & \frac{3}{6} & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 1/6
\end{bmatrix}
\end{array}$$

$$P^2 = \frac{1}{36} \begin{bmatrix}
1 & 3 & 5 & 7 & 9 & 11 \\
0 & 4 & 5 & 7 & 9 & 11 \\
0 & 0 & 9 & 7 & 9 & 11 \\
0 & 0 & 0 & 16 & 9 & 11 \\
0 & 0 & 0 & 0 & 25 & 11 \\
0 & 0 & 0 & 0 & 0 & 36
\end{bmatrix}$$

Initial state probability distribution.

$$p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

(since all the values of 1, 2, 3, 4, 5, 6 are equally likely).

(iii) $$P(x_2 = 6) = \sum_{i=1}^{6} P(x_2 = 6 / x_0 = i)\, P(x_0 = i)$$

$$= \cancel{P(x_2 = 6 / x_0 = 1)\, P(x_0 = 1) +}$$

$$= \frac{1}{6} \sum_{i=1}^{6} P_{i6}^{(2)}$$

$$= \frac{1}{6}\left[P_{16}^{(2)} + P_{26}^{(2)} + P_{36}^{(2)} + P_{46}^{(2)} + P_{56}^{(2)} + P_{66}^{(2)}\right]$$

$$= \frac{1}{6}\left[\frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{36}{36}\right]$$

$$= \cancel{11 + 11 + 11 + 11 + 11 + 36}$$

$$\frac{1}{6 \times 36}\left[11 + 11 + 11 + 11 + 36\right]$$

$$= \frac{91}{216}$$

$=$

$1/16,$

$1/6$

(10)

An urn initially 5 black balls and 5 white balls. the following experiment is repeated indefinitely.
A ball is drawn from the urn.
if the ball is white it is put back in the urn. otherwise it is left out.
let x_n be the number of black balls remaining in the urn after n draws from the urn. Is x_n a markov process, if so find the appropriate transition probabilities
(a) find the one-step t.p.m p for x_n.
(b) Find the two-step T.P.M p by matrix multiplication.
(c) Find the

sol:- the x_n of black balls in the urn

$$P[x_n=4/x_{n-1}=5] = \frac{5C_1}{10C_1} = 5/10$$

x_{n-1} = total 5 B.B
x_n - prob of out coming

$= 1 - P[x_n=5/x_{n-1}=5]$ if not taking black balls

(if black state will change)

$$P[x_n=3/x_{n-1}=4] = \frac{4}{9} = 1 - P[x_n=4/x_{n-1}=4]$$

5B+5W = 5/10
4B+5W = 4/9

$$P[x_n=2/x_{n-1}=3] = \frac{3}{8} = 1 - P[x_n=3/x_{n-1}=3]$$ 3B+5W = 3/8

$$P[x_n=1/x_{n-1}=2] = \frac{2}{7} = 1 - P[x_n=2/x_{n-1}=2]$$

$$P[x_n=0/x_{n-1}=1] = 1/6 = 1 - P[x_n=1/x_{n-1}=1]$$

$P[x_n=0/x_{n-1}=0] = 1$ only white ball coming

(Transition : change of 1 state to another state)

TPM

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2/7 & 5/7 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3/8 & 5/8 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4/9 & 5/9 & 0 \\ 5 & 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{5}{10} \end{array}$$

$P^2 = P * P$

$$P^2 = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11/36 & 25/36 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{21} & \frac{65}{144} & \frac{25}{49} & 0 & 0 & 0 \\ 3 & 0 & 3/28 & \frac{225}{448} & 25/64 & 0 & 0 \\ 4 & 0 & 0 & \frac{1}{6} & \frac{95}{192} & \frac{25}{81} & 0 \\ 5 & 0 & 0 & 0 & 2/9 & 19/36 & 1/4 \end{array}$$

P^2 = 2 step Transition probability

(5) A gambler has a ₹2. He bets 1 at a time and wins rupee 1 with a prob $\frac{1}{2}$. He stops playing if he loses 2 or wins 4 Rs

(a) what is the transition p.m of the related Markov chain?

(b) what is the prob that he has lost his money at the end of 5 plays?

(c) what is the prob that the game lasts for more than 7 plays?

Sol:- Let x_n represent the amount with the player at the end of the n^{th} round of the play.

The state space of $x_n = \{0, 1, 2, 3, 4, 5, 6\}$

when the game is stopped, if the player loses 2, $x_n = 0$ & if he wins 4, $x_n = 6$ $(4+2=6)$

The TPM is

$$P = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 2 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 3 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

0 and 6 states are called absorbing states

If entered he cannot change any other state. (he stops playing)

Initial probability distribution

$p^{(0)} = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)$

$p^{(1)} = p^{(0)} P$

$= (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0) P$

$= (0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 0 \quad 0 \quad 0)$

$p^{(2)} = p^{(1)} P$

$= (0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad 0 \quad 0 \quad 0) P$

$p^{(2)} = (\frac{1}{4} \quad 0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{4} \quad 0 \quad 0)$

$p^{(3)} = p^{(2)} P$

$= (\frac{1}{4} \quad \frac{1}{4} \quad 0 \quad 3/8 \quad 0 \quad 1/8 \quad 0)$

$p^{(4)} = p^{(3)} \cdot P = (3/8 \quad 0 \quad 5/16 \quad 0 \quad 1/4 \quad 0 \quad 1/16)$

$p^{(5)} = p^{(4)} P = (3/8 \quad 5/32 \quad 0 \quad 9/32 \quad 0 \quad 1/8 \quad 1/6)$

$n = 0$

$\cancel{p^{(6)}}$ $P(X_5 = 0) = 3/8$

$p(6) = p(5)\, P$

$= \left(\frac{29}{64} \quad 0 \quad \frac{7}{32} \quad 0 \quad \frac{13}{64} \quad 0 \quad \frac{1}{8} \right)$

$p(7) = p(6) \cdot P$

$= \left(\underset{x=0}{\frac{29}{64}} \quad \underbrace{\frac{7}{64} \quad 0 \quad \frac{27}{128} \quad 0 \quad \frac{13}{128}} \quad \underset{x=6}{\frac{1}{8}} \right)$

$P[X_7 = 1, 2, 3, 4 \,\&\, 5]$

$\therefore \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128}$

$= \frac{27}{64}$

Imp

(3) Three boys A, B & C are throwing a ball to each other. A always throws the ball to B & B always throws the ball to C; but C is just as likely to throw the ball to B as A. Show that the process is markovean. Find the transition matrix and classify the states. Do all the states are ergodic.

sol :- The transition probability matrix of the process $\{x_n\}$ is given below.

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

states of x_n (feature)

$P =$ states of x_{n-1} (present)

$$P = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

States of x_n depends only on states of x_{n-1} but not on states of x_{n-2}, x_{n-3}, ... or earlier states

$\therefore \{x_n\}$ is a markov chain.

Now $p^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$, $p^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

$P_{11}^{(3)} > 0$, $P_{13}^{(2)} > 0$, $P_{21}^{(2)} > 0$, $P_{22}^{(2)} > 0$, $P_{33}^{(2)} > 0$ and all others

$\therefore$ The chain is irreducible

$p^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & 1/2 \\ \frac{1}{4} & 1/2 & 1/4 \end{bmatrix}$, $p^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$

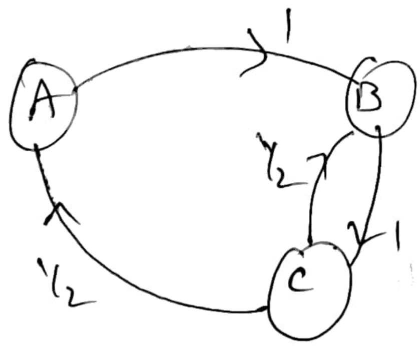
(n)

Sol :- the transition probability matrix of the process $\{x_n\}$ is given below

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \text{ states of } x_n \text{ (future)} \\ \begin{array}{c} \text{states of } x_{n-1} \text{ (present)} \end{array} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

state of X_n depends only on states of X_{n-1} but not on states of $X_{n-2}, X_{n-3}, \ldots$ or earlier states

$\therefore \{x_n\}$ is a markov chain



$x_0 : A \to B$
$x_1 : B \to C$
$x_2 : C \to A$
$x_3 : A \to B$
$x_4 : B \to C$
depends only on previous states.

$A \to B \to C \to A$ so, A is recurrent (come back)

B is "

C " "

$\therefore$ All the states are Recurrent

$d(A) = \text{Gcd}\{3,5\} = 1$

$d(B) = 3, 5$

$\text{Gcd}(3,5) = 1$

$d(C) = 2, 3$

$\text{Gcd}\{2,3\} = 1$

All are communicate to each other so irreducible

A, B, C are aperiodic

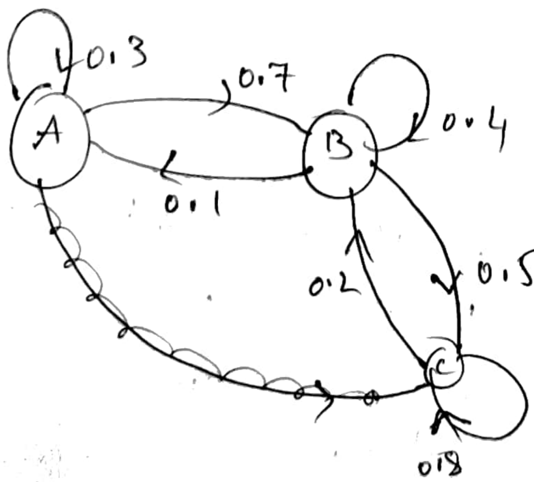
so all are ergodic

(2) The transition probability matrix of a markov chain is given by $\begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{pmatrix}$ is this irreducible (tpm).

sol:-

$$\begin{array}{c|c} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

(Irreducible means: communicate to each other)



∴ It is irreducible

$$p^6 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{8} & 3/8 \end{bmatrix} \text{ \& so on}$$

we note that $P_{ii}^{(2)}, P_{ii}^{(3)}, P_{ii}^{(4)}, P_{ii}^{(5)}, P_{ii}^{(6)}$ etc are > 0 for $i = 2, 3$.

G.c.d of $2, 3, 4, 5, 6 \ldots = 1$

the states 2 & 3 ~~are~~ (i.e B & C) are periodic with period 1

i.e aperiodic.

we note that $P_{ii}^{(3)}, P_{ii}^{(5)}, P_{ii}^{(6)}$ e.t.c are > 0 &

G.c.d of $3, 5, 6 \ldots = 1$

$\therefore$ the state 1 (i.e state A) is periodic with period i.e aperiodic. and irreducible, since the chain is finite all ~~sts~~ its states are non-null persistant.

$\therefore$ All the states are ergodic.

(4) If the matrix
$$\begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
irreducible.

Sol:-

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = P.P = \begin{bmatrix} 0.34 & 0.66 & 0 & 0 \\ 0.33 & 0.67 & 0 & 0 \\ 0.22 & 0.44 & 0.01 & 0.33 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.334 & 0.666 & 0 & 0 \\ 0.333 & 0.667 & 0 & 0 \\ 0.222 & 0.444 & 0.001 & 0.003 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

w.k.T one of the rule to prove that a matrix is irreducible is that the sum of each row of the matrix must be equal to 1, but in matrix P^3, The sum of third row is not equal to 1

Hence the given matrix is not irreducible

The TPM of a markov chain $\{x_n\}$ $n = 1, 2, 3 \ldots$ having 3 states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$

find (i) $P\{x_2 = 3\}$ (ii) $P\{x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2\}$.

sol:- G.T the TPM of markov chain is

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

we have $P(x_0 = 1) = 0.7$, $P(x_0 = 2) = 0.2$

$P(x_0 = 3) = 0.1$

$$P^2 = P * P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

(i) $P(x_2 = 3) = \sum_{i=1}^{3} P\{x_2 = 3 / x_0 = i\} \; P(x_0 = i)$

$= \sum_{i=1}^{3} P_{i3}^{(2)} \; P(x_0 = i)$

$= P_{13}^{(2)} P(x_0 = 1) + P_{23}^{(2)} P(x_0 = 2) + P_{33}^{(2)} P(x_0 = 3)$

$= 0.26(0.7) + 0.34(0.2) + 0.29(0.1)$

$= 0.279$

(ii) $P\{x_3 = 2, x_2 = 3, x_1 = 3, x_0 = 2\}$

$P(x_3 = 2 / x_2 = 3) \; P(x_2 = 3 / x_1 = 3) \; P(x_1 = 3 / x_0 = 2) \; P(x_0 = 2)$

$P_{32}^{(1)} \cdot P_{33}^{(1)} \cdot P_{23}^{(1)} \; 0.2$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= 0.0048$

Period :

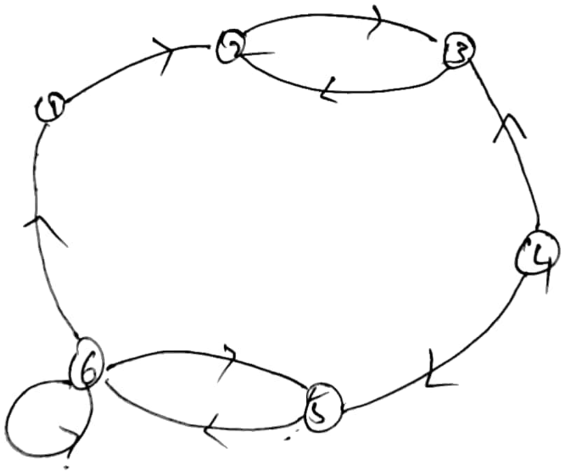
The period of state i is the GCD of n such that $P_{ii}^{(n)} \neq 0$ denoted by $d(i)$

If $d(i) > 1$, we say that the state i is periodic

If $d(i) = 1$, we say that the state i is aperiodic

Note : If $i \leftrightarrow j$ then $d(i) = d(j)$
communicate

Eg :-



$d(1) = 1$ (1 step) $\quad d(4) = 1$

$d(2) = 2 \quad d(5) = 2, 3$, Gcd $(2,3) = 1$

$d(3) = 2 \quad d(6) = 2, 3$ Gcd $(2,3) = 1$

1, 4, 5, 6 - aperiodic

2, 3, are periodic

Equilibrium vector of a Markov chain (16)

If a Markov chain with transition matrix P is regular, then there is a unique vector V such that, for any probability vector v and for large values of n, $\boxed{vP^n = V}$. vector V is called the equilibrium vector or fixed vector or steady state vector of the Markov chain. This is also called long range trend of the Markov chain.

probability vector :- A probability vector is a matrix of only one row, having non-negative entries, with the sum of the entries equal to 1

i.e $V = v_1 + v_2 \; v_n = 1$

① Find the long trend or steady state vector for $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}_{2\times 2}$

sol:- take $vP = v$

$$[v_1 \; v_2] \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix} = [v_1 \; v_2]$$

$$0.25 + 0.5 v_2 = v_1$$

$$0.75 v_1 + 0.5 v_2 = v_2$$

$-0.75v_1 + 0.5v_2 = 0 \rightarrow (1)$

$0.75v_1 + (0.5)v_2 = 0 \rightarrow (2)$

Since v is a equilibrium vector we must have

$$v_1 + v_2 = 1 \rightarrow (3)$$

$(3) \times 0.75$

$$-0.75v_1 + 0.5v_2 = 0$$
$$0.75v_1 + 0.75v_2 = 0.75$$

$$v_1 = 0.25, \quad v_2 = 0.75$$

$$v = [v_1 \ v_2]$$
$$= [0.25 \quad 0.75]$$

(2) Find the long range trend of steady state vector for the markov chain with transition matrix

$$\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Sol :- All the entries are +ve, so this is regular matrix.

$$P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Let v the probability vector $v = [v_1 \; v_2 \; v_3]$, we want to find v such that $vP = v$

$$[v_1 v_2 v_3] \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = [v_1 v_2 v_3]$$

$$0.6v_1 + 0.15v_2 + 0.12v_3 = v_1$$
$$0.28v_1 + 0.67v_2 + 0.36v_3 = v_2$$
$$0.07v_1 + 0.18v_2 + 0.52v_3 = v_3$$

$$-0.35v_1 + 0.15v_2 + 0.12v_3 = 0 \longrightarrow (1)$$
$$0.28v_1 - 0.33v_2 + 0.36v_3 = 0 \longrightarrow (2)$$
$$0.07v_1 + 0.18v_2 - 0.48v_3 = 0 \longrightarrow (3)$$

Since v is the probability vector we have $v_1 + v_2 + v_3 = 1 \longrightarrow (4)$

solve above eqns we get

$$v_1 = \frac{104}{363}, \quad v_2 = \frac{532}{1089}, \quad v_3 = \frac{245}{1089}$$

Steady state vector (or) equilibrium vector

$$v = [v_1 \; v_2 \; v_3] = [0.2865 \;\; 0.4885 \;\; 0.2250]$$

A fair die is tossed repeatedly. If x_n denotes the maximum number occuring in the first n tosses, find the transition probability matrix of the markov chain $\{x_n\}$. Find also p^2.

sol :- Suppose Max number $x_n = 3$ (say)

$x_{n+1} = 3$, $(n+1)^{th}$ state is 1 or 2 or 3

$x_{n+1} = 4$ " " " 4

$x_{n+1} = 5$ " " " 5

$x_{n+1} = 6$ " " " 6

$$p\{x_{n+1} = 3/x_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$p\{x_{n+1} = 4/x_n = 3\} = \frac{1}{6}$$

$$p\{x_{n+1} = 5/x_n = 3\} = \frac{1}{6}$$

$$p\{x_{n+1} = 6/x_n = 3\} = \frac{1}{6}$$

$\rightarrow x_{n+1}$ state

$$\underset{\downarrow x_n}{P} = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

(1) The weather in certain spot is classified as fair, cloudy (without rain) or rainy. A fair day is followed by a fair day 60% of the time and by a cloudy day 25% of the time. A cloudy day is followed by a cloudy day 35% of the time and by a rainy day 25% of the time. A rainy day is followed by a cloudy day 40% of the time and by a rainy day 25% of the time. Initial probabilities are 0.3, 0.3 and 0.4. Find the probability that there will be rainy day after 3 days. What portion of the days is expected to be fair, cloudy or rainy in the long run.

Sol:- We write TPM as

$$P = \begin{array}{c} \\ \text{Fair} \\ \text{cloudy} \\ \text{Rainy} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Fair} & \text{cloudy} & \text{Rainy} \end{array} \\ \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix} \end{array}$$

Initial probabilities are given by

$\pi^0 = [0.3 \quad 0.3 \quad 0.4]$ say

TPM for day 2 = $\pi p = \pi^0 p$

$$\begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0.34 & 0.22 \end{bmatrix}$$

TPM for day 3 $= \pi^2 p$

$$\pi^2 p = \pi p \cdot p$$

$$= \begin{bmatrix} 0.44 & 0.34 & 0.22 \end{bmatrix} \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 0.477 & 0.317 & 0.206 \end{bmatrix}$$

After day 3 = day 4 $= \pi p^3 \Rightarrow \pi p^2 \cdot p$

$$= \begin{bmatrix} 0.477 & 0.317 & 0.206 \end{bmatrix} \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4851 & 0.3126 & 0.2023 \end{bmatrix}$$

After long time, proportion of the days expected to be fair, cloudy or rainy is obtained by using the limiting probability that $\pi_i = i = 0, 1, 2$ are obtained by solving the set of equations (20)

$$\pi_0 = 0.6\pi_0 + 0.25\pi_1 + 0.15\pi_2 \rightarrow (1)$$

$$\pi_1 = 0.4\pi_0 + 0.35\pi_1 + 0.25\pi_2 \rightarrow (2)$$

$$\pi_2 = 0.35\pi_0 + 0.40\pi_1 + 0.25\pi_2 \rightarrow (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \rightarrow (4)$$

solving (1), (2), (3) & (4) we get

$\pi_0 = 0.33$, $\pi_1 = 0.33$, $\pi_2 = 0.33$

$P(x_{t_n} = x_n \mid x(t_{n-1}) = x_{n-1})$ dependency previous event

$\boxed{\text{Stochastic}} \rightarrow \boxed{\text{Markov process}}$

Stochastic → family of R.V

Markov process → $\boxed{\text{Matrix Representation}}$ 1, 2, -- n step.

Markov process → $\boxed{\text{Graphic representation}}$ Graphs

Graphs → $\boxed{\text{Accesible}}$: $i \rightarrow j$, $j \rightarrow i$

$i \leftrightarrow j$

communicate to each other

Accesible → $\boxed{\text{Communicate}}$

transition ~~state~~ prob : $i \rightarrow j$

stochastic :

Regular :

$\boxed{\text{Recurrent}}$ $\boxed{\text{Transiant state}}$

Recurrent: $i \rightarrow j$, $i \leftarrow j$

Transiant state: $i \rightarrow j$, $i \not\leftarrow j$

Note :- Any state, Recurrent or transiant

period : $d(i) = 1$, aperiodic (length of the walk is called period)

$d(i) > 1$, periodic

$\gcd\{n : P_{ii} \neq 0\}$ denoted by $d(i)$

Ergodic : recurrent & aperiodic . then ergodic

irreducible : All the states communicate to each other